Interaction of 3 solitons for the GKdV-4 equation

Georgy A. Omel'yanov[*]

Abstract

We describe an approach to construct multi-soliton asymptotic solutions for non-integrable equations. The general idea is realized in the case of three waves and for the KdV-type equation with nonlinearity u^4. A brief review of asymptotic methods as well as results of numerical simulation are included.

Key words: generalized Korteweg-de Vries equation, soliton, interaction, weak asymptotics method, weak solution, non-integrability

2010 Mathematics Subject Classification: 35Q53, 35D30

1 Introduction

1.1 Statement of the problem

It is well known that an arbitrary number of solitary waves collide for integrable non-linear equations in an enormous manner: they pass through each other almost as linear waves. The aim of this paper is the consideration: can we believe that such type of interaction is conserved (in a sense) for some essentially non-integrable equations?

Needless to recall that the integrability implies both the possibility to find exact solutions of complicated structures and that the equation has some special properties. Conversely, the non-integrability implies that we

---

[*]University of Sonora, Rosales y Encinas s/n, 83000, Hermosillo, Sonora, Mexico, omel@hades.mat.uson.mx

do not have, at least nowadays, neither explicit solutions, nor special useful properties of the problem.

We consider the general techniques and averaging method by a specific but typical example of the Generalized Korteweg-de Vries-4 equation with small dispersion, that is:

$$\frac{\partial u}{\partial t} + \frac{\partial u^m}{\partial x} + \varepsilon^2 \frac{\partial^3 u}{\partial x^3} = 0, \quad m = 4, \quad x \in \mathbb{R}^1, \quad t > 0, \tag{1}$$

where $\varepsilon << 1$ is a small parameter.

The GKDV family (1) contains integrable ($m = 2$ and $m = 3$, that is the KdV and MKdV equations) and essentially non-integrable ($m \geq 4$) equations. The last means that there is not any method at present to construct exact solutions of the Cauchy problem with more or less general initial data.

More in detail, there is known that (1) for $m \geq 6$ is unstable, whereas the case $m = 5$ is conditionally stable (unstable for solitons) [1, 2]. As for $m = 4$, this case is stable and there are known some exact particular solutions including the solitary wave:

$$u = A\omega\left(\beta\frac{x - Vt}{\varepsilon}\right), \quad \omega(\eta) = c\cosh^{-2/3}(\eta), \quad A = \frac{1}{\gamma}\beta^{2/3}, \tag{2}$$

where $\beta > 0$ is an arbitrary number, c is such that

$$\int_{-\infty}^{\infty} \omega(\eta)d\eta = 1, \tag{3}$$

and

$$V = \frac{a_4}{\gamma^3}\beta^2. \tag{4}$$

Here and in what follows we use the notation

$$a_k = \int_{-\infty}^{\infty} \omega^k(\eta)\, d\eta, \quad k \geq 1, \quad a_2' = \int_{-\infty}^{\infty} \left(\frac{d\omega}{d\eta}\right)^2 d\eta \tag{5}$$

and the identities

$$\gamma = \left(\frac{3}{7}\frac{a_2 a_4}{a_2'}\right)^{1/3}, \quad 2a_2 a_4 = \frac{7}{5}a_5. \tag{6}$$

In view of the general wave propagation theory, there appear questions both about the stability of the solitary wave solution (2) with respect to small perturbations of the equation, and about the character of the solitary wave collision.

2

1.2 Prehistory: single-phase asymptotic solutions

The non-integrability implies the use of asymptotic approaches. We consider waves of arbitrary amplitude (of the value $O(1)$), treating the dispersion ε as a small parameter. Thus, there appear "fast" x/ε, t/ε and "slow" x, t variables. Of course, it is possible to rescale $\eta = x/\varepsilon$, $\tau = t/\varepsilon$ and pass to "fast" η, τ and "slow" $\varepsilon\eta$, $\varepsilon\tau$ variables. However we prefer the first version.

The modern asymptotic technique, which is based on ideas by Poincare, van der Pol, Krylov-Bogoliubov and others, has been created firstly by G.E. Kuzmak (ODE, 1959, [3]) and G.B. Whitham, (PDE, 1965, [4, 5], see also [6]) for rapidly oscillating asymptotic solutions of non-linear equations. The famous Whitham method deals with a Lagrangian formulation and allows to find slowly varying amplitude and wave number of non-uniform wave trains. This approach determined the development of the non-linear perturbation theory in 1970s. At the same time, the passage from the original equation to the Lagrangian seemed to be artificial. For this reason J.C. Luke (1966, [7]) created a version of the Whitham method, which allows to construct asymptotic solutions with arbitrary precision appealing directly to the original equation. More in detail, for the equation $L(u, \varepsilon u_t, \varepsilon u_x, \dots) = 0$ we write the ansatz

$$u = Y_0(\tau, t, x) + \varepsilon Y_1(\tau, t, x) + \dots, \tag{7}$$

where $\tau = S(x,t)/\varepsilon$, $Y_k(\tau + T, t, x) = Y_k(\tau, t, x)$, $T = \text{const}$, and $S(x,t)$, $Y_k(\tau, t, x)$ are arbitrary functions from C^∞. Since

$$\varepsilon\partial_t Y_k\big(S(x,t)/\varepsilon, t, x\big) = \{S_t\partial_\tau Y_k(\tau, t, x) + \varepsilon\partial_t Y_k(\tau, t, x)\}|_{\tau=S/\varepsilon},$$

we obtain the chain of ordinary (with respect to τ) equations, the first of them is non-linear,

$$L\big(Y_0(\tau, t, x), S_t\partial_\tau Y_0(\tau, t, x), S_x\partial_\tau Y_0(\tau, t, x), \dots\big) = 0, \tag{8}$$

and others are non-homogenous linearization:

$$L'\big(Y_0(\tau, t, x), S_t\partial_\tau, S_x\partial_\tau, \dots\big) Y_k(\tau, t, x)$$
$$= F_k\big(Y_0(\tau, t, x), \dots, Y_{k-1}(\tau, t, x)\big), \quad k \geq 1. \tag{9}$$

It is assumed that (8) has a T-periodic solution. Then there appear the orthogonality conditions

$$\int_0^T F_k\big(Y_0(\tau, t, x), \dots, Y_{k-1}(\tau, t, x)\big) Z_i d\tau = 0, \quad i = 1, \dots, l, \tag{10}$$

3

which guarantee the solvability of (9) in the space of T-periodic smooth bounded functions. Here $\{Z_i, i = 1, \ldots, l\}$ is the kernel of the operator adjoint to L'. Moreover, (10) allow to define the phase $S(x,t)$ and all the "constants" of integration of the equations (8), (9).

It seemed that the same procedure can be used to construct a perturbed soliton-type solution (with trivial alterations). However, it is not true, and a mechanical repetition of the Whitham construction leads to some "paradoxes" and senseless solutions (see, for example, [8], pp. 303 - 306). The situation has been improved by V. Maslov and G. Omel'yanov (1981, [9], see also [10]). A little bit later a similar construction has been developed by I. Molotkov and S. Vakulenko (see e.g. [11]). To illustrate the modification [9] let us consider the perturbed GKdV-4 equation (1),

$$\frac{\partial u}{\partial t} + \frac{\partial u^4}{\partial x} + \varepsilon^2 \frac{\partial^3 u}{\partial x^3} = R, \tag{11}$$

where $R = R(x, t, u, \varepsilon u_x, \varepsilon^2 u_{xx}, \ldots)$ is "small" in our scaling and $R|_{u=0} = 0$.

To find a self-similar soliton-type asymptotics we restrict the soliton part of the solution on the zero-level set of the phase $S(x,t) = x - \varphi(t) + O((x - \varphi(t))^2)$. This allows to avoid the appearance of some nonuniqueness effects (see [10], pp. 24 - 26). Next we take into account that a smooth small "tail" can appear after the soliton. Therefore, instead of (7) we write the ansatz in the form:

$$u = Y_0(\tau, t) + \varepsilon Y_1(\tau, t, x) + \ldots, \tag{12}$$

where $\tau = (x - \varphi(t))/\varepsilon$, Y_k are smooth bounded function such that $Y_0(\tau, t, x)$ tends to 0 as $\tau \to \pm\infty$, $Y_k(\tau, t, x) \to 0$ as $\tau \to +\infty$, and $Y_k(\tau, t, x) \to Y_k^-(x, t)$ as $\tau \to -\infty$ for $k \geq 1$, and φ belongs to C^∞.

Similar to (8), substituting (12) into (11) we obtain the nonlinear model equation

$$-\varphi_t \frac{dY_0}{d\tau} + \frac{dY_0^4}{d\tau} + \frac{d^3 Y_0}{d\tau^3} = 0 \tag{13}$$

and define the shape of the leading term in (12),

$$Y_0 = A(t)\omega\big(\beta\tau + \varphi_1(t)\big), \quad \beta^{2/3}(t) = \gamma A(t), \tag{14}$$

as well as the similar to (4) relation

$$\frac{d\varphi}{dt} = a_4 A^3(t). \tag{15}$$

4

Here $\varphi_1(t)$ is a "constant" of integration. Next to find the deficient relation between φ and A we consider the first correction Y_1 freezed on the soliton front $x = \varphi(t)$. Denoting $\check{Y}_1(\tau, t) = Y_1(\tau, t, x)|_{x=\varphi(t)}$, we pass to the equation:

$$\frac{d}{d\tau}\left\{ - \varphi_t \check{Y}_1 + 4Y_0^3 \check{Y}_1 + \frac{d^2 \check{Y}_1}{d\tau^2} \right\} = R\big(\varphi, t, Y_0, Y_{0\tau}, Y_{0\tau\tau}, \dots \big) - Y_{0t}. \qquad (16)$$

Respectively, to guarantee the existence of the desired correction $\check{Y}_1$, we obtain the following conditions:

$$\frac{d}{dt} \int_{-\infty}^{\infty} Y_0^2 d\tau = 2 \int_{-\infty}^{\infty} Y_0 R\big(\varphi, t, Y_0, Y_{0\tau}, Y_{0\tau\tau}, \dots \big) d\tau, \qquad (17)$$

$$\varphi_t \check{Y}_1|_{\tau \to -\infty} = \int_{-\infty}^{\infty} \left\{ R\big(\varphi, t, Y_0, Y_{0\tau}, Y_{0\tau\tau}, \dots \big) - Y_{0t} \right\} d\tau. \qquad (18)$$

Calculating the integrals in (17), we complete (15) by the equation

$$a_2 \frac{d}{dt} \frac{A^2}{\beta} = 2 \frac{A}{\beta} \mathfrak{R}, \qquad (19)$$

where

$$\mathfrak{R} = \int_{-\infty}^{\infty} \omega(\eta) R\big(\varphi, t, A\omega(\eta), A\beta\omega(\eta)_\eta, A\beta^2\omega(\eta)_{\eta\eta}, \dots \big) d\eta. \qquad (20)$$

This allows us to determine the phase and amplitude dynamics. The equations (15), (19) have been called "Hugoniot-type conditions" [9] since they do not depend on ε, whereas the solitary wave (14) $Y_0\big((x - \varphi(t))/\varepsilon, t\big)$ disappears (in $\mathcal{D}'$ sense) as $\varepsilon \to 0$. Let us recall that the Rankine-Huginiot conditions remain the same both for parabolic regularization of shock waves, and for the limiting non-smooth solutions.

Furthermore, returning to the asymptotic construction, we note that (18) implies the equality

$$a_4 A^3 \check{Y}_1^- = \frac{1}{\beta} \int_{-\infty}^{\infty} R\big(\varphi, t, A\omega(\eta), A\beta\omega(\eta)_\eta, A\beta^2\omega(\eta)_{\eta\eta}, \dots \big) d\eta - \frac{d}{dt} \frac{A}{\beta}, \qquad (21)$$

where $\check{Y}_1^- = \check{Y}_1|_{\tau \to -\infty}$. Now we integrate the equation (16) and find the structure of the first freezed correction

$$\check{Y}_1(\tau, t) = \check{Y}_1^-(t)\chi(\tau, t) + Z_1(\tau, t) + c_1(t)Y_{0\tau}'(\tau, t), \qquad (22)$$

5

where χ and Z_1 are some fixed functions such that

$$Z_1 \to 0 \quad \text{as} \quad \tau \to \pm\infty,$$
$$\chi \to 0 \quad \text{as} \quad \tau \to +\infty, \quad \chi \to 1 \quad \text{as} \quad \tau \to -\infty,$$

and c_1 is an arbitrary "constant" of integration.

The next step of the construction is the extension of $\check{Y}_1(\tau, t)$ to $Y_1(\tau, t, x)$ in the following manner:

$$Y_1(\tau, t, x) = u_1^-(t, x)\chi(\tau, t) + Z_1(\tau, t) + c_1(t)Y'_{0\tau}(\tau, t), \tag{23}$$

where u_1^- is a smooth function such that

$$\frac{\partial u_1^-}{\partial t} = u_1^- R'_u(x, t, 0, \dots), \quad x < \varphi(t), \quad t > 0, \tag{24}$$

$$u_1^-|_{x=\varphi(t)} = \check{Y}_1^-, \quad t > 0. \tag{25}$$

Continuing the procedure we can easily construct the one-phase self-similar asymptotic solution with arbitrary precision.

Let us note finally that self-similarity implies the special choice of the initial data. In particular, the initial function $Y_1(\tau, 0, x)$ should be of the special form (23) with arbitrary $c_1(0)$ and arbitrary $u_1^-(x, 0)$ under the condition

$$u_1^-(x, 0)|_{x=\varphi(0)} = \check{Y}_1^-(0). \tag{26}$$

If it is violated and, for example, $u|_{t=0} = A(0)\omega\big(\beta(x - \varphi(0))/\varepsilon\big)$, then the perturbed soliton generates a rapidly oscillating tail of the amplitude $o(1)$ (the so called "radiation") instead of the smooth tail $\varepsilon u_1^-(x, t)$ (see [12] for the perturbed KdV equation). However, $\varepsilon u_1^-(x, t)$ describes sufficiently well the tendency of the radiation amplitude behavior (see e.g. [13]).

1.3 History: two-phase asymptotic solution

Concerning the solitary wave collision, this problem is much more complicated. Indeed, to describe the interaction of two waves of the form (2), we should look for the asymptotics as a two-phase function,

$$u = W\left(\frac{x - \varphi_1}{\varepsilon}, \frac{x - \varphi_2}{\varepsilon}, t\right) + o(1), \tag{27}$$

6

where $W(\tau_1, \tau_2, t)$ has properties similar to the two-soliton solution of the KdV equation. However, to construct W we obtain a non-linear PDE, which is, in fact, equivalent to the original GKdV-4 equation (1). So, the existence of such asymptotics remains unknown. The same is true for any essentially non-integrable equation. Respectively, there is not any possibility to construct a classical asymptotic solution (that is, with the remainder in the C-sense).

At the same time it is easy to note that the solitary wave solutions (soliton or kink type) tend to distributions as $\varepsilon \to 0$. This allows to treat the equation in the weak sense and, respectively, look for singularities instead of regular functions. Obviously, non-integrability implies that we cannot find neither classical nor weak exact solutions. However, we can construct an asymptotic weak solution considering the smallness of the remainder in the weak sense.

Originally, such idea had been suggested by V. Danilov and V. Shelkovich for shock wave type solutions (1997, [14]), and after that it has been developed and adapted for many other problems (V. Danilov, G. Omel'yanov, V. Shelkovich, D. Mitrovic and others, [15] - [27] and references therein). We called this approach the "weak asymptotics method".

For the special case of soliton-type solutions we note now that they have the value $O(\varepsilon)$ in the weak sense. Thus, the remainder for the leading term of the asymptotic solution should be $O(\varepsilon^2)$ in the weak sense. However, the GKdV equations (1) degenerate to a first-order PDE in $\mathcal{D}'$ for this precision. The same fact has been noted by Danilov, Omel'yanov, and Radkevich (1997, [28]) by the consideration of a free boundary problem. There has been suggested also a way about how to overcome this obstacle. Applying these ideas to the equation (1), we pass to the following definition of the weak asymptotic solution [17]:

Definition 1. *A sequence $u(t, x, \varepsilon)$, belonging to $\mathcal{C}^\infty(0, T; \mathcal{C}^\infty(\mathbb{R}^1_x))$ for $\varepsilon = \mathrm{const} > 0$ and belonging to $\mathcal{C}(0, T; \mathcal{D}'(\mathbb{R}^1_x))$ uniformly in $\varepsilon \geq 0$, is called a weak asymptotic mod $O_{\mathcal{D}'}(\varepsilon^2)$ solution of (1) if the relations*

$$\frac{d}{dt} \int_{-\infty}^{\infty} u\psi \, dx - \int_{-\infty}^{\infty} u^4 \frac{\partial \psi}{\partial x} dx = O(\varepsilon^2), \tag{28}$$

$$\frac{d}{dt} \int_{-\infty}^{\infty} u^2 \psi \, dx - \frac{8}{5} \int_{-\infty}^{\infty} u^5 \frac{\partial \psi}{\partial x} dx + 3 \int_{-\infty}^{\infty} \left(\varepsilon \frac{\partial u}{\partial x} \right)^2 \frac{\partial \psi}{\partial x} dx = O(\varepsilon^2) \tag{29}$$

hold uniformly in t for any test function $\psi = \psi(x) \in \mathcal{D}(\mathbb{R}^1)$.

7

Here the right-hand sides are $\mathcal{C}^\infty$-functions for $\varepsilon = \text{const} > 0$ and piece-wise continuous functions uniformly in $\varepsilon \geq 0$. The estimates are understood in the $\mathcal{C}(0,T)$ sense:

$$g(t,\varepsilon) = O(\varepsilon^k) \leftrightarrow \max_{t \in [0,T]} |g(t,\varepsilon)| \leq c\varepsilon^k.$$

Definition 2. *A function $v(t,x,\varepsilon)$ is said to be of the value $O_{\mathcal{D}'}(\varepsilon^k)$ if the relation*

$$\int_{-\infty}^{\infty} v(t,x,\varepsilon)\psi(x)dx = O(\varepsilon^k)$$

holds uniformly in t for any test function $\psi \in \mathcal{D}(\mathbb{R}_x^1)$.

The sense of the relation (28) is obvious: it is the adaptation of the standard $\mathcal{D}'$-definition to asymptotic mod $O_{\mathcal{D}'}(\varepsilon^2)$ solution which belongs to $\mathcal{C}(0,T;\mathcal{D}'(\mathbb{R}_x^1))$. Next we note again that (28) cannot be a unique satisfactory condition since here has been lost the difference between the GKdV-4 equation and the limiting first order equation (with $\varepsilon = 0$). To involve the dispersion term into the consideration, we supplement (28) by the additional condition (29). It can be treated as a version of (28) but for special test functions $u\,\psi(x)$, $\psi \in \mathcal{D}(\mathbb{R}_x^1)$, which vary rapidly together with the solution.

It is important also that (29) duplicates the orthogonality condition which appears for single-phase asymptotics. Indeed, the adaptation of the Definition 1 to the perturbed equation (11) implies the transformation of (29) to the following form:

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2\psi dx - \frac{8}{5}\int_{-\infty}^{\infty} u^5\frac{\partial \psi}{\partial x}dx + 3\int_{-\infty}^{\infty}\left(\varepsilon\frac{\partial u}{\partial x}\right)^2\frac{\partial \psi}{\partial x}dx \qquad (30)$$
$$- 2\int_{-\infty}^{\infty} uR\psi dx = O(\varepsilon^2).$$

Next for u of the form (12), (14) we calculate the weak expansion:

$$\int_{-\infty}^{\infty} u^k(x,t)\psi(x)dx = \varepsilon\frac{A^k}{\beta}\int_{-\infty}^{\infty}\omega^k(\eta)\psi(\varphi + \varepsilon\frac{\eta}{\beta})dx + O(\varepsilon^2)$$
$$= \varepsilon a_k\frac{A^k}{\beta}\psi(\varphi) + O(\varepsilon^2),$$

where $k \geq 1$. Thus

$$u^k = \varepsilon a_k\frac{A^k}{\beta}\delta(x - \varphi) + O_{\mathcal{D}'}(\varepsilon^2). \qquad (31)$$

8

In the same manner we obtain

$$(\varepsilon u_x)^2 = \varepsilon a_2' \beta A^2 \delta(x - \varphi) + O_{\mathcal{D}'}(\varepsilon^2), \tag{32}$$

$$uR(x, t, u, \varepsilon u_x, \varepsilon^2 u_{xx}, \dots) = \varepsilon \frac{A}{\beta} \Re \delta(x - \varphi) + O_{\mathcal{D}'}(\varepsilon^2), \tag{33}$$

where $\Re$ has been defined in (20). Substitution of (31) - (33) into (30) implies the relation

$$\varepsilon \left\{ -a_2 \frac{A^2}{\beta} \frac{d\varphi}{dt} + a_5 \frac{8}{5} \frac{A^5}{\beta} - 3a_2' \beta A^2 \right\} \delta'(x - \varphi)$$
$$+ \varepsilon \left\{ a_2 \frac{d}{dt} \frac{A^2}{\beta} - 2\frac{A}{\beta} \Re \right\} \delta(x - \varphi) = O_{\mathcal{D}'}(\varepsilon^2). \tag{34}$$

Since $\delta(x - \varphi)$ and $\delta'(x - \varphi)$ are linearly independent, their coefficients in (34) should be equal to zero. Taking into account the identities (4) we obtain again the equations (15), (19) for the one-phase asymptotics (12).

Let us revert to the two-wave interaction. Following [17] (see also [18]), we present the ansatz as the sum of two distorted solitons (2), that is:

$$u = \sum_{i=1}^{2} G_i \omega \left(\beta_i \frac{x - \varphi_i}{\varepsilon} \right), \tag{35}$$

where

$$G_i = A_i + S_i(\tau), \quad \varphi_i = \varphi_{i0}(t) + \varepsilon \varphi_{i1}(\tau), \quad \tau = \beta_1 \big(\varphi_{20}(t) - \varphi_{10}(t) \big) / \varepsilon, \tag{36}$$

A_i are the original amplitudes and $\varphi_{i0} = V_i t + x_{i0}$ describe the trajectories of the non-interacting waves (2), $\beta_i = (\gamma A_i)^{2/3}$. We assume that $A_1 < A_2$ and $x_{10} >> x_{20}$, therefore, the trajectories $x = \varphi_{10}$ and $x = \varphi_{20}$ intersect at a point (x^*, t^*). Next we define the "fast time" τ to characterize the distance between the trajectories φ_{i0} and we assume that $S_i(\tau), \varphi_{i1}(\tau)$ are such that

$$S_i \to 0 \quad \text{as} \quad \tau \to \pm\infty, \tag{37}$$

$$\varphi_{i1} \to 0 \quad \text{as} \quad \tau \to -\infty, \quad \varphi_{i1} \to \varphi_{i1}^\infty = \text{const}_i \quad \text{as} \quad \tau \to +\infty. \tag{38}$$

It is obvious that the existence of the weak asymptotics (35) with the properties (37), (38) implies that the solitary waves (2) interact like the KdV solitons at least in the leading term.

9

To construct the asymptotics we should calculate again the weak expansions for u^k and $(\varepsilon u_x)^2$. It is easy to check that

$$u = \varepsilon \sum_{i=1}^{2} \frac{G_i}{\beta_i} \delta(x - \varphi_i) + O_{\mathcal{D}'}(\varepsilon^3). \tag{39}$$

At the same time

$$\int_{-\infty}^{\infty} u^2(x,t)\psi(x)dx = \varepsilon \sum_{i=1}^{2} \frac{G_i^2}{\beta_i} \int_{-\infty}^{\infty} \omega^2(\eta)\psi(\varphi_i + \varepsilon\frac{\eta}{\beta_i})dx \tag{40}$$

$$+ 2G_1 G_2 \int_{-\infty}^{\infty} \omega\left(\beta_1 \frac{x - \varphi_1}{\varepsilon}\right) \omega\left(\beta_2 \frac{x - \varphi_2}{\varepsilon}\right) \psi(x)dx. \tag{41}$$

We take into account that the integrand in (41) vanishes exponentially fast as $|\varphi_1 - \varphi_2|$ grows, thus, the main contribution gives the point x^*. We write

$$\varphi_{i0} = x^* + V_i(t - t^*) = x^* + \varepsilon\frac{V_i}{\beta_1(V_2 - V_1)}\tau \quad \text{and} \quad \varphi_i = x^* + \varepsilon\chi_i, \tag{42}$$

where $\chi_i = V_i\tau/\left(\beta_1(V_2 - V_1)\right) + \varphi_{i1}$. Next we transform the integral in (41) to the following form:

$$\frac{\varepsilon}{\beta_2} \int_{-\infty}^{\infty} \omega(\theta_{12}\eta - \sigma_{12})\omega(\eta)\psi\left(x^* + \varepsilon\chi_2 + \varepsilon\frac{\eta}{\beta_2}\right)d\eta, \tag{43}$$

where $\theta_{12} = \beta_1/\beta_2$, $\sigma_{12} = \beta_1(\varphi_1 - \varphi_2)/\varepsilon$. It remains to apply the formula

$$f(\tau)\delta(x - \varphi_i) = f(\tau)\delta(x - x^*) - \varepsilon\chi_i f(\tau)\delta'(x - x^*) + O_{\mathcal{D}'}(\varepsilon^2), \tag{44}$$

which holds for each φ_i of the form (42) with slowly increasing χ_i and for $f(\tau)$ from the Schwartz space. Moreover, the second term in (44) is $O_{\mathcal{D}'}(\varepsilon)$. Thus, under the assumptions (37) we can modify (39)-(41) to the final form:

$$u = \varepsilon \sum_{i=1}^{2} \frac{A_i}{\beta_i}\delta(x - \varphi_i) + \varepsilon \sum_{i=1}^{2} \frac{S_i}{\beta_i}\left\{\delta(x - x^*) - \varepsilon\chi_i\delta'(x - x^*)\right\} + O_{\mathcal{D}'}(\varepsilon^3), \tag{45}$$

$$u^2 = \varepsilon a_2 \sum_{i=1}^{2} \frac{A_i^2}{\beta_i}\delta(x - \varphi_i) + \varepsilon a_2 \left\{\sum_{i=1}^{2} \frac{1}{\beta_i}(2A_iS_i + S_i^2)\right.$$

10

$$+ 2\frac{G_1 G_2}{\beta_2}\lambda_{2,1}(\sigma_{12})\Big\}\delta(x - x^*) + O_{\mathcal{D}'}(\varepsilon^2), \tag{46}$$

where the convolution $\lambda_{2,1}(\sigma_{12})$ describes the product of two waves. In view of further applications we present such type of convolutions in the general version:

$$\lambda_{m,k}^{(j)}(\sigma_{ln}) = \frac{1}{a_m}\int_{-\infty}^{\infty}\eta^j\omega^{m-k}(\eta_{ln})\omega^k(\eta)d\eta, \ \eta_{ln} \stackrel{\mathrm{def}}{=} \theta_{ln}\eta - \sigma_{ln}, \ \theta_{ln} \stackrel{\mathrm{def}}{=} \frac{\beta_l}{\beta_n}, \tag{47}$$

where $1 \le k < m$, $m \ge 2$, $j = 0$ or $j = 1$, and we write $\lambda_{m,k}(\sigma_{ln}) \stackrel{\mathrm{def}}{=} \lambda_{m,k}^{(0)}(\sigma_{ln})$ simplifying the notation.

To calculate the time-derivative of u with the accuracy $O_{\mathcal{D}'}(\varepsilon^2)$ it is enough to use the expansion (45), the assumptions (37), (38), and to apply the formula (44) again. Thus,

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{2}\frac{\dot\psi_0}{\beta_i}\frac{dS_i}{d\tau}\delta(x - x^*) - \varepsilon\sum_{i=1}^{2}V_i\frac{A_i}{\beta_i}\delta'(x - \varphi_i)$$

$$- \varepsilon\dot\psi_0\frac{d}{d\tau}\sum_{i=1}^{2}\Big\{\frac{A_i}{\beta_i}\varphi_{i1} + \chi_i K_{i1}^{(1)}\Big\}\delta'(x - x^*) + O_{\mathcal{D}'}(\varepsilon^2), \tag{48}$$

where $\dot\psi_0 = \beta_1(V_2 - V_1)$.

On the contrary, to find $\partial(u^2)/\partial t$ with the same accuracy we should add to the leading term (46) the next correction:

$$- \varepsilon^2 a_2\Big\{\sum_{i=1}^{2}\frac{\chi_i}{\beta_i}(2A_i S_i + S_i^2)$$

$$+ 2\frac{G_1 G_2}{\beta_2}\Big(\chi_2\lambda_{2,1}(\sigma_{12}) + \frac{1}{\beta_2}\lambda_{2,1}^{(1)}(\sigma_{12})\Big)\Big\}\delta'(x - x^*) + O_{\mathcal{D}'}(\varepsilon^3). \tag{49}$$

Now we obtain

$$\frac{\partial u^2}{\partial t} = a_2\frac{d}{d\tau}\Big\{2\frac{\dot\psi_0}{\beta_2}G_1 G_2\lambda_{2,1}(\sigma_{12}) + \sum_{i=1}^{2}\frac{\dot\psi_0}{\beta_i}(2A_i S_i + S_i^2)\Big\}\delta(x - x^*)$$

$$- \varepsilon a_2\frac{d}{d\tau}\Big\{\dot\psi_0\sum_{i=1}^{2}\frac{A_i^2}{\beta_i}\varphi_{i1} + 2\frac{\dot\psi_0}{\beta_2}G_1 G_2\big(\chi_2\lambda_{2,1}(\sigma_{12}) + \frac{1}{\beta_2}\lambda_{2,1}^{(1)}(\sigma_{12})\big) \tag{50}$$

11

$$+\sum_{i=1}^{2}\frac{\dot{\psi}_0}{\beta_i}\chi_i\big(2A_iS_i+S_i^2\big)\bigg\}\delta'(x-x^*)-\varepsilon a_2\sum_{i=1}^{2}V_i\frac{A_i^2}{\beta_i}\delta'(x-\varphi_i)+O_{\mathcal{D}'}(\varepsilon^2).$$

Calculating weak expansions for the other terms from the left-hand sides of (28), (29) and substituting them into (28), (29) we obtain linear combinations of $\delta'(x-\varphi_i)$, $i=1,2$, $\delta(x-x^*)$, and $\delta'(x-x^*)$. Therefore, we pass to the following system of 8 equations for 8 unknowns:

$$P_{i,j}(A_i,\beta_i,V_i)=0,\quad j=1,2,\quad i=1,2, \tag{51}$$

$$\frac{d}{d\tau}Q_j(S_1,S_2,\sigma_{12})=0,\quad j=1,2, \tag{52}$$

$$\frac{d\varphi_{j1}}{d\tau}=R_j(S_1,S_2,\sigma_{12}),\quad j=1,2. \tag{53}$$

The first four algebraic equations (51) imply again the relations (4) between A_i, β_i, and V_i. Furthermore, functional equations (52) allow to define S_i with the property (37), whereas an analysis of the ODE (53) justifies the existence of the required phase corrections φ_{i1} with the property (38). By analogy with (15), (19) we call (51)-(53) the Hugoniot-type conditions again. Results of
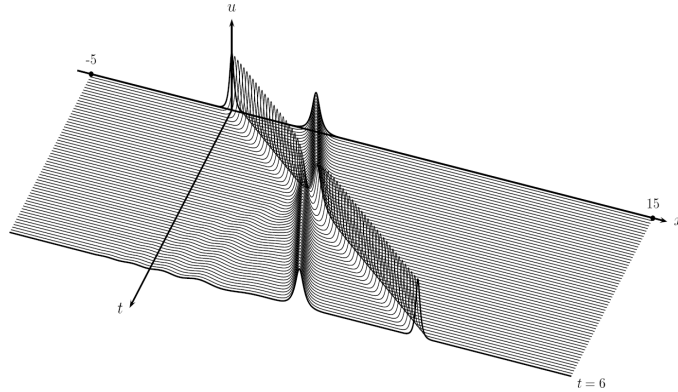

Fig. 1: Evolution of two solitary waves for $\varepsilon=0.1$

numerical simulations [29, 30] confirm the traced asymptotic analysis (see Figure 1).

Finally let us note that the two-wave problem for the equation (1) has been considered recently in the framework of another approach but for small amplitudes [31].

1.4 The next problem: three wave interaction

Further numerical investigation of the GKdV-4 equation showed that N solitary waves collide elastically (in the leading term) at least for $N \leq 5$, see [29, 30] and Figures 2–4. Thus, there appears the problem of describing the
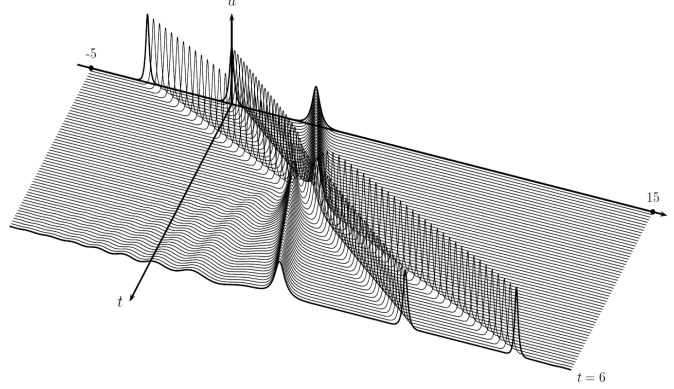

Fig. 2: Evolution of the soliton triplet for $\varepsilon = 0.1$

interaction of three (and more) solitary waves. It seemed that it was enough to repeat the same procedure as above, now for three waves. However, it is easy to recognize that the corresponding 4 equations (52), (53) will contain now 6 free functions. Obviously, this can not be any adequate description of the solution.

Therefore, we should transform the conception of the asymptotic solution. To do it let us recall two first conservation laws for the equation (1) (in the differential form):

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left\{u^4 + \varepsilon^2 \frac{\partial^2 u}{\partial x^2}\right\} = 0, \tag{54}$$

$$\frac{\partial u^2}{\partial t} + \frac{\partial}{\partial x}\left\{\frac{8}{5}u^5 - 3\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2 + \varepsilon^2\frac{\partial^2 u^2}{\partial x^2}\right\} = 0. \tag{55}$$

Comparing the left-hand sides of (54), (55) with (28), (29) we conclude that Definition 1 calls a function to be a "weak asymptotic solution" if it satisfies the conservation laws (54), (55) in the sense $O_{\mathcal{D}'}(\varepsilon^2)$. Next note that perturbations of (1) imply corresponding perturbations of the conservation laws. For (11), instead of (54), (55), we have the relations

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left\{u^4 + \varepsilon^2\frac{\partial^2 u}{\partial x^2}\right\} = R, \tag{56}$$
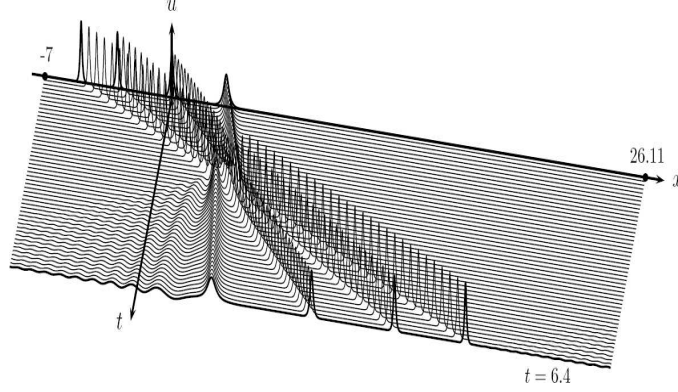
13

Fig. 3: Evolution of 4 solitons for $\varepsilon = 0.1$

$$\frac{\partial u^2}{\partial t} + \frac{\partial}{\partial x}\left\{\frac{8}{5}u^5 - 3\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2 + \varepsilon^2\frac{\partial^2 u^2}{\partial x^2}\right\} = 2Ru. \qquad (57)$$

Reverting to the single-phase asymptotic solution (12) one can easily establish that the orthogonality condition (17), and therefore the equation (19), is the integral form of (57), calculated for (12) with the accuracy $O_{\mathcal{D}'}(\varepsilon^2)$. At the same time (18), and thus the equality (21), is the integral form of (56) $\mod O_{\mathcal{D}'}(\varepsilon^2)$, calculated for (12), where Y_1 has the form (23) and u_1^- satisfies the equation (24). Note also that the second "conservation law"
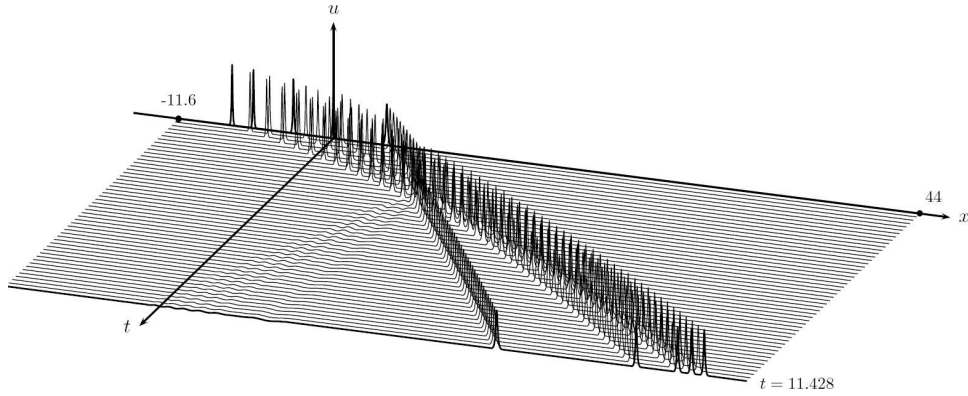

Fig. 4: Evolution of 5 solitons for $\varepsilon = 0.1$

(57) has been used in the one-phase situation to define the principal term of the asymptotic solution, whereas (56) has been used only to define the first correction Y_1.

14

Therefore, we see that to define the principal asymptotics term there has been used only one conservation law for the single-phase solution, and two conservation laws for the two-phase solution. So it is natural to assume that to construct a three-phase asymptotics we should add to (54), (55) the third conservation law, namely

$$\frac{\partial}{\partial t}\Big\{\Big(\varepsilon\frac{\partial u}{\partial x}\Big)^2 - \frac{2}{5}u^5\Big\} - \frac{\partial}{\partial x}\Big\{2\varepsilon^2\frac{\partial u}{\partial t}\frac{\partial u}{\partial x} + \Big(u^4 + \varepsilon^2\frac{\partial^2 u}{\partial x^2}\Big)^2\Big\} = 0. \qquad (58)$$

2 Asymptotic construction

Let us consider the equation (1) with the Cauchy data

$$u|_{t=0} = \sum_{i=1}^{3} A_i\omega\Big(\beta_i\frac{x - x_{i0}}{\varepsilon}\Big), \qquad (59)$$

where $A_1 < A_2 < A_3$, $x_{10} >> x_{20} >> x_{30}$.

The arguments considered in the previous subsection imply the following

Definition 3. *A sequence $u(t, x, \varepsilon)$, belonging to $\mathcal{C}^\infty(0, T; \mathcal{C}^\infty(\mathbb{R}^1_x))$ for $\varepsilon = $ const > 0 and belonging to $\mathcal{C}(0, T; \mathcal{D}'(\mathbb{R}^1_x))$ uniformly in ε, is called the weak asymptotic mod $O_{\mathcal{D}'}(\varepsilon^2)$ solution of the problem (1), (59) if the relations (28), (29), and*

$$\frac{\partial}{\partial t}\Big\{\int_{-\infty}^{\infty}\Big(\varepsilon\frac{\partial u}{\partial x}\Big)^2\psi dx - \frac{2}{5}\int_{-\infty}^{\infty}u^5\psi dx\Big\} \qquad (60)$$

$$+ \int_{-\infty}^{\infty}\Big\{2\varepsilon^2\frac{\partial u}{\partial t}\frac{\partial u}{\partial x} + u^8 + \Big(\varepsilon^2\frac{\partial^2 u}{\partial x^2}\Big)^2 + 2u^4\varepsilon^2\frac{\partial^2 u}{\partial x^2}\Big\}\frac{\partial\psi}{\partial x}dx = O(\varepsilon^2)$$

hold uniformly in t for any test function $\psi = \psi(x) \in \mathcal{D}(\mathbb{R}^1)$.

To construct the asymptotic solution we write the ansatz in the form similar to (35), namely

$$u = \sum_{i=1}^{3} G_i\omega\Big(\beta_i\frac{x - \varphi_i}{\varepsilon}\Big), \qquad (61)$$

where the same notation and hypothesis (36)-(38) are assumed with obvious corrections: the "fast time" is defined now using the distance between the first and third trajectories,

$$\tau = \beta_1\big(\varphi_{30}(t) - \varphi_{10}(t)\big)/\varepsilon, \qquad (62)$$

15

and we suppose the intersection of all trajectories $x = \varphi_{i0}(t)$, $i = 1, 2, 3$, at the same point (x^*, t^*).

The technic of our approach has been explained in Subsection 1.3. So we clarify here some new detail only using as an example u^m. All others explicit formulas for the asymptotic expansions are presented in the Appendix.

Lemma 1. *Let the assumptions (37), (38) for u of the form (61) be satisfied. Then the following asymptotic expansions hold:*

$$u^m = \varepsilon a_m \left\{ \sum_{i=1}^{3} K_{i0}^{(m)} \delta(x - \varphi_i) + \mathfrak{R}_m \delta(x - x^*) \right\} + O_{\mathcal{D}'}(\varepsilon^2), \qquad (63)$$

$$\frac{\partial u^m}{\partial t} = a_m \dot{\psi}_0 \frac{d\mathfrak{R}_m}{d\tau} \delta(x - x^*) - \varepsilon a_m \sum_{i=1}^{3} V_i K_{i0}^{(m)} \delta'(x - \varphi_i)$$

$$- \varepsilon a_m \dot{\psi}_0 \frac{d}{d\tau} \left\{ \sum_{i=1}^{3} K_{i0}^{(m)} \varphi_{i1} + \mathfrak{R}_m^{(1)} \right\} \delta'(x - x^*) + O_{\mathcal{D}'}(\varepsilon^2), \qquad (64)$$

where $m \geq 1$,

$$\dot{\psi}_0 = \beta_1(V_3 - V_1), \ K_i^{(m)} = \frac{G_i^m}{\beta_i}, \ K_{i0}^{(m)} = \frac{A_i^m}{\beta_i}, \ K_{i1}^{(m)} = K_i^{(m)} - K_{i0}^{(m)}, \quad (65)$$

$$\mathfrak{R}_m = \sum_{i=1}^{3} K_{i1}^{(m)} + \sum_{l,n} R_{m,ln} + R_{m,123}, \qquad (66)$$

$$\mathfrak{R}_m^{(1)} = \sum_{i=1}^{3} \chi_i K_{i1}^{(m)} + \sum_{l,n}(\chi_n R_{m,ln} + C_{m,ln}) + \chi_3 R_{m,123} + C_{m,123}, \qquad (67)$$

$$R_{m,ln} = \beta_l \sum_{k=1}^{m-1} C_m^k K_l^{(m-k)} K_n^{(k)} \lambda_{m,k}(\sigma_{ln}), \qquad (68)$$

$$R_{m,123} = \beta_1 \beta_2 \sum_{j=2}^{m-1} \sum_{k=1}^{j-1} C_m^j C_j^k K_1^{(m-j)} K_2^{(j-k)} K_3^{(k)} \lambda_{m,123}^{(0),(j,k)}, \qquad (69)$$

$$C_{m,ln} = \sum_{k=1}^{m-1} C_m^k \theta_{ln} K_l^{(m-k)} K_n^{(k)} \lambda_{m,k}^{(1)}(\sigma_{ln}), \qquad (70)$$

$$C_{m,123} = \beta_1 \theta_{23} \sum_{j=2}^{m-1} \sum_{k=1}^{j-1} C_m^j C_j^k K_1^{(m-j)} K_2^{(j-k)} K_3^{(k)} \lambda_{m,123}^{(1),(j,k)}, \qquad (71)$$

16

C_m^k are the binomial coefficients, the notation (47) has been used, and

$$\sum_{l,n} f_{ln} \stackrel{\text{def}}{=} f_{12} + f_{13} + f_{23}, \quad \sigma_{ln} = \beta_l \frac{\varphi_l - \varphi_n}{\varepsilon}, \quad \chi_i = \frac{V_i}{\dot{\psi}_0}\tau + \varphi_{i1}. \quad (72)$$

Furthermore,

$$\lambda_{m,123}^{(i),(j,k)} = \frac{1}{a_m} \int_{-\infty}^{\infty} \eta^i \omega^{m-j}(\eta_{13})\omega^{j-k}(\eta_{23})\omega^k(\eta)d\eta, \quad i = 0 \quad or \quad i = 1. \quad (73)$$

To prove the lemma let us separate all the terms of u^m into three groups: one-phase, two-phase and three-phase functions:

$$u^m = \sum_{i=1}^{3} Y_i^m + \sum_{l,n}\sum_{k=1}^{m-1} C_m^k Y_l^{m-k}Y_n^k + \sum_{j=1}^{m-1}\sum_{k=1}^{j-1} C_m^j C_j^k Y_1^{m-j}Y_2^{j-k}Y_3^k, \quad (74)$$

where $Y_i = G_i\omega\big(\beta_i(x - \varphi_i)/\varepsilon\big)$. Now considering u^m in the weak sense we change the variable: $x = \varphi_i + \varepsilon\eta/\beta_i$, $x = \varphi_n + \varepsilon\eta/\beta_n$, and $x = \varphi_3 + \varepsilon\eta/\beta_3$ respectively for the integrals of the groups. Next, preparing the same transformations as in Subsection 1.3 and applying (44) again we pass to the formula (63).

In the same manner one can prove the following proposition:

Lemma 2. *Let the assumptions (37), (38) be satisfied for u of the form (61). Then the following asymptotic expansions hold:*

$$(\varepsilon u_x)^2 = \varepsilon a_2'\left\{ \sum_{i=1}^{3} \beta_i^2 K_{i0}^{(2)}\delta(x - \varphi_i) + \Re_{(1),2}\delta(x - x^*)\right\} + O_{\mathcal{D}'}(\varepsilon^2), \quad (75)$$

$$\frac{\partial}{\partial t}(\varepsilon u_x)^2 = a_2'\dot{\psi}_0\frac{d\Re_{(1),2}}{d\tau}\delta(x - x^*) - \varepsilon a_2'\sum_{i=1}^{3}\beta_i^2 V_i K_{i0}^{(2)}\delta'(x - \varphi_i)$$

$$- \varepsilon a_2'\dot{\psi}_0\frac{d}{d\tau}\left\{ \sum_{i=1}^{3} \beta_i^2 K_{i0}^{(2)}\varphi_{i1} + \Re_{(1),2}^{(1)}\right\}\delta'(x - x^*) + O_{\mathcal{D}'}(\varepsilon^2), \quad (76)$$

$$(\varepsilon^2 u_{xx})^2 = \varepsilon a_2''\left\{ \sum_{i=1}^{3} \beta_i^4 K_{i0}^{(2)}\delta(x - \varphi_i) + \Re_{(2),2}\delta(x - x^*)\right\} + O_{\mathcal{D}'}(\varepsilon^2), \quad (77)$$

$$\varepsilon^2 u^4 u_{xx} = \varepsilon a_{23}\left\{ -4\sum_{i=1}^{3} \beta_i^2 K_{i0}^{(5)}\delta(x - \varphi_i) + \mathfrak{L}\delta(x - x^*)\right\} + O_{\mathcal{D}'}(\varepsilon^2), \quad (78)$$

17

$$\varepsilon^2 u_x u_t = -\varepsilon a_2' \sum_{i=1}^{3} \beta_i^2 V_i K_{i0}^{(2)} \delta(x - \varphi_i) - \varepsilon a_2' \mathfrak{P} \delta(x - x^*) + O_{\mathcal{D}'}(\varepsilon^2), \qquad (79)$$

where a_2', $\dot{\psi}_0$, and $K_{i0}^{(2)}$ are defined in (5), (65), $\mathfrak{P} = \dot{\psi}_0 (\mathfrak{S} + \mathfrak{S}_G) + \mathfrak{M}$, $\mathfrak{S}$, $\mathfrak{S}_G$, $\mathfrak{M}$, and other notation are deciphered in Attachment, Subsections 6.1 and 6.2.

Now we substitute the expansions (63), (64), and (75)-(79) into (28), (29), (60) and obtain the similar (51)-(53) system. Namely, the algebraic system for each $i = 1, 2, 3$:

$$- V_i K_{i0}^{(1)} + a_4 K_{i0}^{(4)} = 0, \qquad (80)$$

$$- a_2 V_i K_{i0}^{(2)} + \frac{8}{5} a_5 K_{i0}^{(5)} - 3 a_2' \beta_i^2 K_{i0}^{(2)} = 0, \qquad (81)$$

$$- a_2' \beta_i^2 V_i K_{i0}^{(2)} + \frac{2}{5} a_5 V_i K_{i0}^{(5)} + 2 a_2' \beta_i^2 V_i K_{i0}^{(2)}$$
$$- a_8 K_{i0}^{(8)} + 8 a_{23} \beta_i^2 K_{i0}^{(5)} - a_2'' \beta_i^4 K_{i0}^{(2)} = 0, \qquad (82)$$

the system of functional equations:

$$\sum_{i=1}^{3} K_{i1}^{(1)} = 0, \qquad (83)$$

$$\mathfrak{R}_2 = 0, \qquad (84)$$

$$a_2' \mathfrak{R}_{(1),2} - \frac{2}{5} a_5 \mathfrak{R}_5 = 0, \qquad (85)$$

and the system of ordinary differential equations:

$$- \dot{\psi}_0 \frac{d}{d\tau} \Big\{ \sum_{i=1}^{3} K_{i0}^{(1)} \varphi_{i1} + \chi_i K_{i1}^{(1)} \Big\} + a_4 \mathfrak{R}_4 = 0, \qquad (86)$$

$$- a_2 \dot{\psi}_0 \frac{d}{d\tau} \Big\{ \sum_{i=1}^{3} K_{i0}^{(2)} \varphi_{i1} + \mathfrak{R}_2^{(1)} \Big\} + \frac{8}{5} a_5 \mathfrak{R}_5 - 3 a_2' \mathfrak{R}_{(1),2} = 0, \qquad (87)$$

$$\dot{\psi}_0 \frac{d}{d\tau} \Big\{ - a_2' \Big(\sum_{i=1}^{3} \beta_i^2 K_{i0}^{(2)} \varphi_{i1} + \mathfrak{R}_{(1),2}^{(1)} \Big) + \frac{2}{5} a_5 \Big(\sum_{i=1}^{3} K_{i0}^{(5)} \varphi_{i1} + \mathfrak{R}_5^{(1)} \Big) \Big\}$$
$$+ 2 a_2' \mathfrak{P} - a_8 \mathfrak{R}_8 - 2 a_{23} \mathfrak{L} - a_2'' \mathfrak{R}_{(2),2} = 0. \qquad (88)$$

Let us overcome the first obstacle: for each i the system (80)-(82) of three equation contains only two free parameters A_i, V_i.

18

Lemma 3. *Let $\omega(\eta)$, $A_i = A(\beta_i)$, and $V_i = V(\beta_i)$ be of the form (2)-(6). Then the equalities (80)-(82) are satisfied uniformly in $\beta_i > 0$.*

Proof. Obviously, equations (80), (81) coincide with (51) and imply again the formulas (2)-(6). Substituting them into (82), we transform it to the following form:

$$a_2 a_4^2 - a_8 - a_2'' \gamma^6 + 8 a_{23} \gamma^3 = 0. \tag{89}$$

Next we note that $\omega(\eta)$ satisfies the model equation

$$\gamma^3 \frac{d^2 \omega}{d\eta^2} = a_4 \omega - \omega^4. \tag{90}$$

Multiplying (90) for ω'' and integrating, we obtain the identity

$$4 a_{23} = \gamma^3 a_2'' + a_4 a_2'. \tag{91}$$

On the other hand, integrating the squares of the left-hand and right-hand parts of (90), we pass to another identity:

$$a_8 = \gamma^6 a_2'' - a_2 a_4^2 + 2 a_4 a_5. \tag{92}$$

This and (6) verify the equality (89). $\qquad\square$

Since the system of six equations (83)-(88) contains six free functions, we obtain the first formal result

Theorem 1. *Let the system (83)-(88) have a solution which satisfies the assumptions of the form (37), (38). Then the solitary waves (61) collide preserving mod $O_{\mathcal{D}'}(\varepsilon^2)$ the KdV-type scenario of interaction.*

Moreover, similar to the Rankine-Hugoniot condition, which is simply the conservation law for the shock-wave solution, the Hugoniot-type conditions (80)-(88) imply the verification of some conservation laws:

Theorem 2. *Let the assumptions of Theorem 1 be satisfied. Then the ansatz (61) is a mod $O_{\mathcal{D}'}(\varepsilon^2)$ asymptotic solution of the equation (1) if and only if (61) satisfies the conservation laws*

$$\frac{d}{dt} \int_{-\infty}^{\infty} u \, dx = 0, \quad \frac{d}{dt} \int_{-\infty}^{\infty} u^2 \, dx = 0, \quad \frac{d}{dt} \int_{-\infty}^{\infty} \left\{ \left(\varepsilon \frac{\partial u}{\partial x} \right)^2 - \frac{2}{5} u^5 \right\} dx = 0, \tag{93}$$

19

and the energy relations

$$\frac{d}{dt} \int_{-\infty}^{\infty} xu\,dx - \int_{-\infty}^{\infty} u^4 dx = 0,$$

$$\frac{d}{dt} \int_{-\infty}^{\infty} xu^2 dx - \frac{8}{5} \int_{-\infty}^{\infty} u^5 dx + 3 \int_{-\infty}^{\infty} \left(\varepsilon \frac{\partial u}{\partial x} \right)^2 = 0, \tag{94}$$

$$\frac{d}{dt} \left\{ \int_{-\infty}^{\infty} x \left(\varepsilon \frac{\partial u}{\partial x} \right)^2 dx - \frac{2}{5} \int_{-\infty}^{\infty} xu^5 dx \right\}$$

$$+ 2\varepsilon^2 \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} \frac{\partial u}{\partial x} dx + \int_{-\infty}^{\infty} \left(u^4 + \varepsilon^2 \frac{\partial^2 u}{\partial x^2} \right)^2 dx = 0.$$

To prove this conclusion it is enough to rewrite the equations (80)-(88) in the integral form.

3 Analysis of the Hugoniot-type conditions

3.1 Transformations

Let us pay the attention to the equations (83)-(85). Normalization

$$\kappa_i = \gamma \beta_3^{1/3} S_i / \beta_i \tag{95}$$

implies

$$K_i^{(m)} = \frac{\beta_3^{2m/3-1}}{\gamma^m} \theta_{i3}^{2m/3-1} \Lambda_i^m, \quad \text{where} \quad \Lambda_i = 1 + \theta_{i3}^{1/3} \kappa_i. \tag{96}$$

We denote $\overline{\mathfrak{R}}_m = \gamma^m \mathfrak{R}_m / \beta_3^{2m/3-1}$ and obtain:

$$\overline{\mathfrak{R}}_m = \sum_{i=1}^{3} \theta_{i3}^{2m/3-1} (\Lambda_i^m - 1) + \sum_{l,n} \overline{R}_{m,ln} + \overline{R}_{m,123} \tag{97}$$

$$\overline{R}_{m,ln} = \sum_{k=1}^{m-1} C_m^k \theta_{l3}^{2(m-k)/3} \theta_{n3}^{2k/3-1} \Lambda_l^{m-k} \Lambda_n^k \lambda_{m,k}(\sigma_{ln}), \tag{98}$$

$$\overline{R}_{m,123} = \sum_{j=2}^{m-1} \sum_{k=1}^{j-1} C_m^j C_j^k \theta_{13}^{2(m-j)/3} \theta_{23}^{2(j-k)/3} \Lambda_1^{m-j} \Lambda_2^{j-k} \Lambda_3^k \lambda_{m,123}^{(0),(j,k)}. \tag{99}$$

This and similar formulas for $\mathfrak{R}_{(1),2} \overset{\text{def}}{=} \beta_3^{7/3}\overline{\mathfrak{R}}_{(1),2}/\gamma^2$ (see Attachment) allow us to transform (83)-(85) to the following form:

$$\sum_{i=1}^{3} \kappa_i = 0, \tag{100}$$

$$\sum_{i=1}^{3} \theta_{i3}^{1/3}(\Lambda_i^{\,2} - 1) + 2\sum_{l,n} \theta_{l3}^{1/3}\theta_{ln}^{1/3}\,\Lambda_l\,\Lambda_n\,\lambda_{2,1}(\sigma_{ln}) = 0, \tag{101}$$

$$\sum_{i=1}^{3} \theta_{i3}^{7/3}(\Lambda_i^{\,2} - 1) + 2\sum_{l,n} \theta_{l3}^{5/3}\theta_{n3}^{2/3}\,\Lambda_l\,\Lambda_n\lambda_{I1}^{(0)}(\sigma_{ln}) - \frac{4}{3}\overline{\mathfrak{R}}_5 = 0, \tag{102}$$

where the equalities (6), the notation (97)-(99), and (140) have been used.

Next let us simplify the equations (86)-(88). We note firstly that in view of (83) and the identity

$$\beta_l(\chi_l - \chi_n) = \sigma_{ln} \tag{103}$$

it is possible to eliminate χ_i from the left-hand side of (86), since

$$\sum_{i=1}^{3} K_{i0}^{(1)}\varphi_{i1} + \chi_i K_{i1}^{(1)} = \sum_{i=1}^{3} K_{i0}^{(1)}\varphi_{i1} + \frac{\sigma_{12}}{\beta_1}K_{11}^{(1)} - \frac{\sigma_{23}}{\beta_2}K_{31}^{(1)}.$$

In the same manner, applying (84) and (85), we simplify the equations (87), (88). Thus, we transform (86)-(88) to the following form:

$$\dot{\psi}_0 \frac{d}{d\tau}\Big\{ \sum_{i=1}^{3} K_{i0}^{(1)}\varphi_{i1} + \frac{\sigma_{12}}{\beta_1}K_{11}^{(1)} - \frac{\sigma_{23}}{\beta_2}K_{31}^{(1)} \Big\} = f, \tag{104}$$

$$\dot{\psi}_0 \frac{d}{d\tau}\Big\{ \sum_{i=1}^{3} K_{i0}^{(2)}\varphi_{i1} + \sum_{l,n} C_{2,ln} + \frac{\sigma_{12}}{\beta_1}K_{11}^{(2)}$$
$$- \frac{\sigma_{23}}{\beta_2}\big(K_{31}^{(2)} + R_{2,13} + R_{2,23}\big) \Big\} = F, \tag{105}$$

$$\dot{\psi}_0 \frac{d}{d\tau}\Big\{ \sum_{i=1}^{3} \Big(\beta_i^2 K_{i0}^{(2)} - \frac{4}{3}\gamma^3 K_{i0}^{(5)}\Big)\varphi_{i1} + \mathfrak{K} \Big\} - 2\mathfrak{S} = \mathfrak{F}, \tag{106}$$

where

$$f = a_4\mathfrak{R}_4, \quad F = \frac{a_2'}{a_2}\mathfrak{R}_{(1),2}, \quad \mathfrak{F} = 2\mathfrak{M} - \frac{a_8}{a_2'}\mathfrak{R}_8 - 2\frac{a_{23}}{a_2'}\mathfrak{L} - \frac{a_2''}{a_2'}\mathfrak{R}_{(2),2}, \tag{107}$$

21

and the function $\mathfrak{K}$ is described in Attachment (see formula (142)).

The second step is the elimination of φ_{i1} from the model system. To do it we divide σ_{ln} into the growing $(\overline{\sigma}_{ln})$ and the bounded $(\tilde{\sigma}_{ln})$ parts:

$$\sigma_{ln} = \overline{\sigma}_{ln} + \tilde{\sigma}_{ln}, \quad \overline{\sigma}_{ln} \stackrel{\text{def}}{=} \frac{\beta_l}{\dot{\psi}_0}(V_l - V_n)\tau \tag{108}$$

and rewrite the identity (103):

$$\tilde{\sigma}_{ln} = \beta_l(\varphi_{l1} - \varphi_{n1}). \tag{109}$$

Thus

$$\varphi_{11} = \frac{\tilde{\sigma}_{12}}{\beta_1} + \varphi_{21}, \quad \varphi_{31} = -\frac{\tilde{\sigma}_{23}}{\beta_2} + \varphi_{21}. \tag{110}$$

Substituting (110) into (104) we obtain

$$\dot{\psi}_0 \frac{d\varphi_{21}}{d\tau} = -\frac{\dot{\psi}_0}{r_1} \frac{d}{d\tau}\left\{ \frac{\tilde{\sigma}_{12}}{\beta_1}K_{10}^{(1)} + \frac{\sigma_{12}}{\beta_1}K_{11}^{(1)} - \frac{\tilde{\sigma}_{23}}{\beta_2}K_{30}^{(1)} - \frac{\sigma_{23}}{\beta_2}K_{31}^{(1)} \right\} + \frac{f}{r_1}. \tag{111}$$

Here and in what follows we use the notation

$$r_j = \sum_{i=1}^{3} K_{i0}^{(j)} \quad \text{for} \quad j = 1 \quad \text{and} \quad j = 2. \tag{112}$$

Next we use the equalities (110), (111), and

$$\tilde{\sigma}_{13} = \tilde{\sigma}_{12} + \theta_{12}\tilde{\sigma}_{23}, \quad \overline{\sigma}_{13} = \overline{\sigma}_{12} + \theta_{12}\overline{\sigma}_{23}, \tag{113}$$

and rewrite (105), (106) as equations for new unknowns $\tilde{\sigma}_{12}$, $\tilde{\sigma}_{23}$. After normalization (95) we pass to the following model equations:

$$\dot{\psi}_0 \frac{d}{d\tau}\left\{ p_{10}\frac{\tilde{\sigma}_{12}}{\beta_1} + p_{11}\frac{\sigma_{12}}{\beta_1} - p_{30}\frac{\tilde{\sigma}_{23}}{\beta_2} - p_{31}\frac{\sigma_{23}}{\beta_2} + \sum_{l,n} C_{2,ln} \right\} = F - \frac{r_2}{r_1}f, \tag{114}$$

$$\dot{\psi}_0 \left\{ \frac{d}{d\tau}\left\{ e_{10}\frac{\tilde{\sigma}_{12}}{\beta_1} + e_{11}\frac{\sigma_{12}}{\beta_1} \right\} + 2\Psi\frac{d}{d\tau}\left\{ K_{11}^{(1)}\frac{\sigma_{12}}{\beta_1} \right\} - \frac{d}{d\tau}\left\{ e_{30}\frac{\tilde{\sigma}_{23}}{\beta_2} + e_{31}\frac{\sigma_{23}}{\beta_2} \right\} \right.$$

$$\left. - 2\Psi\frac{d}{d\tau}\left\{ K_{31}^{(1)}\frac{\sigma_{23}}{\beta_2} \right\} + r_1\frac{d\mathfrak{K}_C}{d\tau} - 2r_1\mathfrak{S}_G \right\} = \mathfrak{F}_1, \tag{115}$$

where $\mathfrak{F}_1 = r_1\mathfrak{F} + (2\Psi + \sum_{i=1}^{3} q_{i0}^{(2)})f$ and the coefficients $p_i^{(k)}$, e_{ki}, $q_{ik}^{(m)}$, Ψ are presented in Attachment (see formulas (144) - (148)).

3.2 Asymptotic analysis

To simplify the further analysis let us assume that

$$\theta_{23}^{1/3} = \mu, \ \theta_{12}^{2/3} = \mu^{1+\alpha}, \text{ where } \alpha \in [0,1) \text{ and } \mu \text{ is sufficiently small.} \quad (116)$$

We look for the asymptotic solution of the system (100)-(102) in the form:

$$\kappa_1 = \frac{1}{2}\mu^\alpha(y_1 - \mu^{2-\alpha}x_0), \quad \kappa_2 = -\frac{1}{2}\mu^\alpha(y_1 + \mu^{2-\alpha}x_0), \quad \kappa_3 = \mu^2 x_0, \quad (117)$$

where x_0 and y_1 are free functions. Then (100) is satisfied, whereas (101) and (102) imply the system:

$$2x_0 - \mu^\alpha y_1 \left\{ 1 + \mu\lambda_{2,1}(\sigma_{23}) - \mu^{1+\alpha}(1 - \lambda_{2,1}(\sigma_{12}) + \frac{1}{4}y_1) - \mu^{3(1+\alpha)/2}\lambda_{2,1}(\sigma_{12}) \right\}$$
$$= -2\lambda_{2,1}(\sigma_{23}) - 2\mu^\alpha\lambda_{2,1}(\sigma_{12}) - 2\mu^{1+\alpha}\lambda_{2,1}(\sigma_{13}) + O_\lambda(\mu^2), \quad (118)$$

$$\left\{ 7 + 40\mu^2\lambda_{5,4}(\sigma_{23}) + \frac{37}{2}\mu^2 x_0 \right\}x_0 - 5\mu^{1+\alpha}y_1\lambda_{5,4}(\sigma_{23})$$
$$= -10\lambda_{5,4}(\sigma_{23}) - 10\mu^{1+\alpha}\lambda_{5,4}(\sigma_{13}) + O_\lambda(\mu^2 + \mu^{(3+7\alpha)/2}). \quad (119)$$

Here and in what follows we denote

$$f(\sigma,\mu) = O_\lambda(\mu^k) \quad \text{if} \quad \max_\sigma |f(\sigma,\mu)| \le c\mu^k \quad (120)$$

and $f(\sigma,\cdot)$ belongs to the Schwartz space.

It is easy to see that the compatibility of the equations (118) and (119) requires the condition: $10\lambda_{5,4}(\sigma_{23}) = 7\lambda_{2,1}(\sigma_{23}) + O_\lambda(\mu^\alpha)$.

Lemma 4. *Let $\omega(\eta)$ be of the form (2). Then*

$$10\lambda_{5,4}(\sigma_{ln}) = 7\lambda_{2,1}(\sigma_{ln}) + 3\theta_{ln}\lambda_{I1}^{(0)}(\sigma_{ln}) \quad (121)$$

for all indices l, n.

To prove the lemma it is enough to use again the equation (68) and the identities (5), (6), (67).

Now we set

$$x_0 = -\lambda_{2,1}(\sigma_{23}) + \mu^\alpha x_1 \quad (122)$$

and transform (118), (119) to the final form:

$$2x_1 - r_{12}y_1 = -2\lambda_{2,1}(\sigma_{12}) - 2\mu\lambda_{2,1}(\sigma_{13}) + O_\lambda(\mu^{2-\alpha}), \quad (123)$$

23

$$r_{21}x_1 - \frac{5}{7}\mu\lambda_{5,4}(\sigma_{23})y_1 = -\mu\lambda_{2,1}(\sigma_{13}) - \frac{3}{7}\mu^{3-\alpha}\lambda_{I1}^{(0)}(\sigma_{23}) + O_\lambda(\mu^3), \quad (124)$$

where

$$r_{12} = 1 + \mu\lambda_{2,1}(\sigma_{23}) - \mu^{1+\alpha}\left(1 - \lambda_{2,1}(\sigma_{12}) + \frac{1}{4}y_1\right) + O_\lambda(\mu^{3(1+\alpha)/2}),$$

$$r_{21} = 1 + \frac{19}{4}\mu^2\lambda_{2,1}(\sigma_{23}) + O_\lambda(\mu^{2+\alpha}).$$

Solving this system we obtain the asymptotic representation:

$$x_1 = -\mu\big(\lambda_{2,1}(\sigma_{13}) - \lambda_{2,1}(\sigma_{12})\lambda_{2,1}(\sigma_{23})\big) + O_\lambda(\mu^{2+\alpha}), \quad (125)$$

$$y_1 = 2\lambda_{2,1}(\sigma_{12})\Big(1 + \mu^{1+\alpha}\big(1 - \lambda_{2,1}(\sigma_{12})\big)\Big) + O_\lambda(\mu^{2-\alpha}). \quad (126)$$

Combining (117), (122), (125), and (126) we conclude:

Lemma 5. *Let there exist functions φ_{i1}, $i = 1, 2, 3$, with the properties (38) and let the condition (116) be realized. Then the system (100) - (102) has the unique solution*

$$\kappa_1 = \mu^\alpha\lambda_{2,1}(\sigma_{12})\big\{1 + \mu^{1+\alpha}\big(1 - \lambda_{2,1}(\sigma_{12})\big)\big\} + O_\lambda(\mu^2), \quad (127)$$

$$\kappa_2 = -\mu^\alpha\lambda_{2,1}(\sigma_{12})\big\{1 + \mu^{1+\alpha}\big(1 - \lambda_{2,1}(\sigma_{12})\big)\big\} + O_\lambda(\mu^2), \quad (128)$$

$$\kappa_3 = -\mu^2\lambda_{2,1}(\sigma_{23}) + O_\lambda(\mu^{3+\alpha}), \quad (129)$$

such that $S_i = \beta_i\kappa_i/\gamma\beta_3^{1/3}$ satisfy the assumptions (37).

To complete the analysis we should prove the solvability of the system (114), (115). Taking into account (95), (116), and (127) - (128), we obtain:

$$\tilde{E}_{11}\frac{d\tilde{\sigma}_{12}}{d\tau} - \theta_{12}\tilde{E}_{12}\frac{d\tilde{\sigma}_{23}}{d\tau} = \tilde{F}_1, \quad (130)$$

$$\tilde{E}_{21}\frac{d\tilde{\sigma}_{12}}{d\tau} - \theta_{12}\tilde{E}_{22}\frac{d\tilde{\sigma}_{23}}{d\tau} = \tilde{F}_2, \quad (131)$$

where the coefficients $\tilde{E}_{ij}$ and right-hand sides $\tilde{F}_i$ are demonstrated in Attachment, Subsection 6.4.

It is easy to calculate that

$$\det(\tilde{E}_{ij}) = \theta_{12}\mu\Delta, \quad (132)$$

24

where

$$\Delta = \frac{7}{3} + O(\mu^{(1+3\alpha)/2}) + O_\lambda(\mu^{(3+\alpha)/2} + \mu^2).$$

Thus, we transform the system (130), (131) to the standard form

$$\frac{d\tilde{\sigma}_{12}}{d\tau} = \tilde{M}_{12}(\tau, \sigma_{12}, \sigma_{23}, \mu)/\Delta, \quad \frac{d\tilde{\sigma}_{23}}{d\tau} = \tilde{M}_{23}(\tau, \sigma_{12}, \sigma_{23}, \mu)/\Delta, \qquad (133)$$

where

$$\tilde{M}_{12} = -\frac{20}{3}\mu z'(\sigma_{23}) + O_\lambda(\mu^{(3+\alpha)/2}), \quad z(\sigma) \stackrel{\text{def}}{=} \sigma\lambda_{2,1}(\sigma), \qquad (134)$$

$$\tilde{M}_{23} = -2\mu^{-\alpha}\lambda_{2,1}(\sigma_{23}) - \frac{7}{3}z'(\sigma_{12}) + O_\lambda(\mu^{(1-\alpha)/2} + \mu^\alpha), \qquad (135)$$

and the equalities (6), (91), (92), as well as the functional relation (121) and

$$a_8\lambda_{8,7}(\sigma_{ln}) = a_2 a_4^4 \lambda_{2,1}(\sigma_{ln}) - \gamma^3 a_{23}\lambda_{4,3(2)}(\sigma_{ln}) + \theta_{ln}\gamma^3 a_4 a_2' \lambda_{I1}^{(0)}(\sigma_{ln}) \qquad (136)$$

have been taken into account.

According to the notation (108) and the first assumption of the form (38) we add to (133) the "initial" condition:

$$\tilde{\sigma}_{12}\big|_{\tau\to-\infty} \to 0, \quad \tilde{\sigma}_{23}\big|_{\tau\to-\infty} \to 0. \qquad (137)$$

Since $\tilde{M}_{ij}$ vanish with an exponential rate as $\tau \to \pm\infty$, it is easy to prove the solvability of the problem (133), (137). Next we note that $\lambda_{2,1}(\sigma_{ln}) = \lambda_{2,1}(\dot{\bar\sigma}_{ln}\tau + \tilde{\sigma}_{ln})$. Since $\dot{\bar\sigma}_{23} = O(\mu^{-3(1+\alpha)/2})$ we find from (133), (135) that $\tilde{\sigma}_{23}(\tau) = O(\mu^{(3+\alpha)/2})$ for sufficiently large τ, however it tends to the limiting value sufficiently slowly, with an exponent $O(\mu^{(3+\alpha)/2})$. Conversely, taking into account that $\dot{\bar\sigma}_{12} = O(\mu^6)$, we obtain that $\tilde{\sigma}_{12}(\tau) = O(1)$ for sufficiently large τ and tends to the limit with an exponent $O(1)$.

The last step of the construction is the return to the phase corrections φ_{i1}. In view of (110), (111) it is obvious that the last assumption of the form (38) is justified. This implies our main proposition

Theorem 3. *Under the assumption (116) the asymptotic solution (61) describes* mod $O_{\mathcal{D}'}(\varepsilon^2)$ *the KdV-type scenario of the solitary waves interaction.*

4 Conclusion

We looked for an approach to describe solitary wave collisions avoiding the use of explicit multi-soliton formulas. Surprisingly, we came back to the ancient Whitham's idea to construct asymptotics with the help of conservation laws and a reasonable ansatz, but in the framework of the weak asymptotics method. In our case three conservation laws for three waves have been utilized. It is clear now how to generalize the approach: for N waves N conservation laws should be used. On contrary, the existence of N conservation laws does not imply the existence of N-soliton type solution since some very astonishing additional conditions appear to guarantee both the solvability of model equations (like (121)) and the regularity of the solutions (like (136)). Furthermore, some questions remain open, the first of them: how to choose the collection of conservation laws to describe N-soliton interaction and is it possible to change conservation laws to reasonable energy relations? At the same time we can formulate the main result of the paper: there is not a sharp frontier between integrable and nonintegrable equations: similar scenarios of the soliton interaction are realized, but with small corrections in the nonintegrable case.

5 Acknowledgement

References

[1] J. L. Bona, P. E. Souganidis and W. Strauss, "Stability and instability of solitary waves of Korteweg-de Vries type", Proc. Roy. Soc. London Ser. A, **411** (1841), 395–412 (1987).

[2] F. Merle, "Existence of blow-up solutions in the energy space for the critical generalized KdV equation", J. Amer. Math. Soc., **14** (3), 555–578 (2001).

[3] G. E. Kuzmak, "Asymptotic solutions of nonlinear second order differential equations with variable coefficients", J. Appl. Math. Mech., **23**, 730–744 (1959).

[4] G. B. Whitham, "Nonlinear dispersive waves", Proc. Roy. Soc. Ser. A, **283**, 238–261, (1965).

[5] G. B. Whitham, "A general approach to linear and non-linear dispersive waves using a Lagrangian", J. Fluid Mech., **22**, 273–283, (1965).

[6] G. B. Whitham, *Linear and nonlinear waves* (Wiley, NY, 1974).

[7] J. C. Luke, "A perturbatiom method for nonlinear dispersive wave problems", Proc. Roy. Soc. London Ser. A, **292**, 403–412, (1966).

[8] A. Scott, *Nonlinear science: emergence and dynamics of coherent structures* (Oxford University Prtess, NY, 1999).

[9] V. P. Maslov and G. A. Omel'yanov, "Asymptotic soliton-form solutions of equations with small dispersion", Uspekhi Mat. Nauk **36**, 63–126, (1981); English transl. in Russian Math. Surveys **36**, 73–149, (1981).

[10] V. P. Maslov and G. A. Omel'yanov, *Geometric Asymptotics for Nonlinear PDE* (AMS, Providence, RI, 2001).

[11] I. A. Molotkov and S. A. Vakulenko, *Concentrated Nonlinear Waves* (Leningrad. Univ., Leningrad, 1988).

[12] L. A. Kalyakin, "Perturbation of the Korteweg-de Vrries soliton", Theoret. and Math. Phys., **92**, 736–747, (1992).

[13] G. A. Omel'yanov and M. A. Valdez-Grijalva, "Asymptotics for a C^1-version of the KdV equation", Nonlinear Phenomena in Complex Systems, **17** (2), 106-115, (2014).

[14] V. G. Danilov and V. M. Shelkovich, "Generalized solutions of nonlinear differential equations and the Maslov algebras of distributions", Integral Transformations and Special Functions, **6**, 137–146, (1997).

[15] V. G. Danilov and V. M. Shelkovich, "Propogation and interaction of shock waves of quasilinear equations", Nonlinear Stadies, **8** (1), 135–169, (2001).

[16] V. G. Danilov and V. M. Shelkovich, "Dynamics of propagation and interaction of delta-shock waves in conservation law systems", Journal of Differential Equations, **211** (2), 333-381, (2005).

27

[17] V. G. Danilov and G. A. Omel'yanov, "Weak asymptotics method and the interaction of infinitely narrow delta-solitons", Nonlinear Analysis: Theory, Methods and Applications, **54**, 773–799, (2003).

[18] V. G. Danilov, G. A. Omel'yanov and V. M. Shelkovich, Weak asymptotics method and interaction of nonlinear waves, in: M.V. Karasev (Ed.), *Asymptotic methods for wave and quantum problems*, AMS Trans., Ser. 2, v. 208, AMS, Providence, RI, pp. 33–164, (2003).

[19] R. F. Espinoza and G. A. Omel'yanov, "Asymptotic behavior for the centered-rarefaction appearance problem", Electron. J. Diff. Eqns., **2005** (148), 1–25, (2005).

[20] D. A. Kulagin and G. A. Omel'yanov, "Interaction of kinks for semilinear wave equations with a small parameter". Nonlinear Analysis, **65** (2), 347–378, (2006).

[21] M. G. Garcia and G. A. Omel'yanov, "Kink-antikink interaction for semilinear wave equations with a small parameter", Electron. J. Diff. Eqns., **2009** (45), 1–26, (2009).

[22] G. A. Omel'yanov, "About the stability problem for strictly hyperbolic systems of conservation laws", Rend. Sem. Mat. Univ. Politec. Torino, **69** (4), 377–392, (2011).

[23] G. A. Omel'yanov and I. Segundo Caballero, "Interaction of solitons for sine-Gordon-type equations", Journal of Mathematics, **2013**, Article ID 845926, 1–8, (2013).

[24] E. Yu. Panov and V. M. Shelkovich, "δ'-shock waves as a new type of solutions to systems of conservation laws", Journal of Differential Equations, **228** (1), 49-86, (2006).

[25] V. G. Danilov and D. Mitrovic, "Shock wave formation process for a multidimensional scalar conservation law", Quart. Appl. Math., **69** (4), 613–634, (2011).

[26] H. Kalisch and D. Mitrovic, "Singular solutions of a fully nonlinear 2×2 system of conservation laws", Proceedings of the Edinburgh Mathematical Society II, **55**, 711–729, (2012).

[27] Xiumei Li and Chun Shen, "Viscous Regularization of Delta Shock Wave Solution for a Simplified Chromatography", System. Abstr. App. Anal., **2013** , Article ID 893465, 1–10, (2013).

[28] V. G. Danilov, G. A. Omel'yanov and E. V. Radkevich, "Weak solutions to the phase field system", Integral Transformations and Special Functions, **6**, 27–35, (1997).

[29] M. G. Garcia and G. A. Omel'yanov, "Interaction of solitary waves for the generalized KdV equation", Communications in Nonlinear Science and Numerical Simulation, **17** (8), 3204–3218, (2012).

[30] M. G. Garcia and G. A. Omel'yanov, "Interaction of solitons and the effect of radiation for the generalized KdV equation", Communications in Nonlinear Science and Numerical Simulation, **19** (8), 2724-2733, (2014).

[31] Y. Martel and F. Merle, "Description of two soliton collision for the quartic gKdV equation", Ann. of Math., **174** (2), 757–857, (2011).

6 Attachment

6.1 Formulas for $(\varepsilon u_x)^2$ and $(\varepsilon^2 u_{xx})^2$

$$\mathfrak{R}_{(k),2} = \sum_{i=1}^{3} \beta_i^{2k} K_{i1}^{(2)} + \sum_{l,n} R_{2,ln}^{(k)}, \ R_{2,ln}^{(k)} = 2\beta_l^{1+k} \beta_n^k K_l^{(1)} K_n^{(1)} \lambda_{I,k}^{(0)}(\sigma_{ln}), \quad (138)$$

$$\mathfrak{R}_{(2),2}^{(1)} = \sum_{i=1}^{3} \beta_i^2 \chi_i K_{i1}^{(2)} + \sum_{l,n}(\chi_n R_{2,ln}^{(1)} + C_{2,ln}^{(1)}), \quad (139)$$

$$C_{2,ln}^{(1)} = 2\beta_l^2 K_l^{(1)} K_n^{(1)} \lambda_{I,1}^{(1)}(\sigma_{ln}), \ \lambda_{I,1}^{(i)}(\sigma_{ln}) = \frac{1}{a_2'}\int_{-\infty}^{\infty} \eta^i \omega'(\eta_{ln})\omega'(\eta)d\eta, \quad (140)$$

$$\lambda_{I,2}(\sigma_{ln}) = \frac{1}{a_2''}\int_{-\infty}^{\infty} \omega''(\eta_{ln})\omega''(\eta)d\eta, \quad a_2'' = \int_{-\infty}^{\infty}\left(\omega''(\eta)\right)^2 d\eta, \quad (141)$$

and $\omega'(\eta) = d\omega(\eta)/d\eta$, $\omega''(\eta) = d^2\omega(\eta)/d\eta^2$.

6.2 Formulas for $\varepsilon^2 u_x u_t$ and $\varepsilon^2 u^4 u_{xx}$

$$\mathfrak{S} = \sum_{i=1}^{3} \beta_i^2 K_i^{(2)} \frac{d\varphi_{i1}}{d\tau} + \sum_{l,n} \beta_l^2 \beta_n K_l^{(1)} K_n^{(1)} \left(\frac{d\varphi_{l1}}{d\tau} + \frac{d\varphi_{n1}}{d\tau} \right) \lambda_{I,1}^{(0)}(\sigma_{ln}),$$

$$\mathfrak{M} = \sum_{i=1}^{3} \beta_i^2 V_i K_{i1}^{(2)} + \sum_{l,n} \mathfrak{M}_{ln}, \ \mathfrak{M}_{ln} = \beta_l^2 \beta_n K_l^{(1)} K_n^{(1)} (V_l + V_n) \lambda_{I,1}^{(0)}(\sigma_{ln}),$$

$$\mathfrak{S}_G = \sum_{l,n} \left(G_l \frac{dG_n}{d\tau} - G_n \frac{dG_l}{d\tau} \right) \lambda_{0I}(\sigma_{ln}), \ \mathfrak{L} = -4 \sum_{i=1}^{3} \beta_i^2 K_{i1}^{(5)} + \sum_{l,n} \mathfrak{L}_{ln} + \mathfrak{Q},$$

$$\mathfrak{L}_{ln} = \sum_{j=1}^{4} C_4^j \beta_l^3 K_l^{(5-j)} K_n^{(j)} \lambda_{4,j(1)}(\sigma_{ln}) + \sum_{j=0}^{3} C_4^j \beta_l \beta_n^2 K_l^{(4-j)} K_n^{(j+1)} \lambda_{4,j(2)}(\sigma_{ln}),$$

$$\mathfrak{Q} = \sum_{j=1}^{3} C_4^j \Big\{ \beta_1 \beta_2 \beta_3^2 K_1^{(4-j)} K_2^{(j)} K_3^{(1)} \lambda_{5,j(1)}$$

$$+ \beta_1 \beta_2^3 K_1^{(4-j)} K_2^{(1)} K_3^{(j)} \lambda_{5,j(2)} + \beta_1^3 \beta_2 K_1^{(1)} K_2^{(4-j)} K_3^{(j)} \lambda_{5,j(3)} \Big\}$$

$$+ \sum_{j=2}^{3} \sum_{k=1}^{j-1} C_4^j C_j^k \beta_1 \beta_2 K_1^{(4-j)} K_2^{(j-k)} K_3^{(k)} \sum_{m=1}^{3} \beta_m^3 K_m^{(1)} \lambda_{6,jkm},$$

$$\lambda_{0I}(\sigma_{ln}) = \frac{1}{a_2'} \int_{-\infty}^{\infty} \omega(\eta_{ln}) \omega'(\eta) d\eta, \quad a_{23} = \int_{-\infty}^{\infty} \omega^3(\eta) \Big(\omega'(\eta) \Big)^2 d\eta,$$

$$\lambda_{4,j(1)}(\sigma_{ln}) = \frac{1}{a_{23}} \int_{-\infty}^{\infty} \omega^{4-j}(\eta_{ln}) \omega^j(\eta) \omega''(\eta_{ln}) d\eta,$$

$$\lambda_{4,j(2)}(\sigma_{ln}) = \frac{1}{a_{23}} \int_{-\infty}^{\infty} \omega^{4-j}(\eta_{ln}) \omega^j(\eta) \omega''(\eta) d\eta,$$

$$\lambda_{5,j(1)} = \frac{1}{a_{23}} \int_{-\infty}^{\infty} \omega^{4-j}(\eta_{13}) \omega^j(\eta_{23}) \omega''(\eta) d\eta,$$

$$\lambda_{5,j(2)} = \frac{1}{a_{23}} \int_{-\infty}^{\infty} \omega^{4-j}(\eta_{13}) \omega''(\eta_{23}) \omega^j(\eta) d\eta,$$

$$\lambda_{5,j(3)} = \frac{1}{a_{23}} \int_{-\infty}^{\infty} \omega''(\eta_{13}) \omega^{4-j}(\eta_{23}) \omega^j(\eta) d\eta,$$

$$\lambda_{6,jkm} = \frac{1}{a_{23}} \int_{-\infty}^{\infty} \omega^{4-j}(\eta_{13}) \omega^{j-k}(\eta_{23}) \omega^k(\eta) \omega''(\eta_{m3}) d\eta.$$

6.3 Normalization

$$\overline{\mathfrak{R}}_{(1),2} = \sum_{i=1}^{3} \theta_{i3}^{7/3}(\Lambda_i^2 - 1) + 2\sum_{l,n} \theta_{l3}^{5/3}\theta_{n3}^{2/3}\,\Lambda_l\,\Lambda_n\lambda_{I,1}^{(0)}(\sigma_{ln}),$$

$$\mathfrak{K} = \frac{\sigma_{12}}{\beta_1}q_{11}^{(1)} - \frac{\sigma_{23}}{\beta_2}\left(q_{31}^{(1)} + Q_3\right) + \mathfrak{K}_C, \; q_{ik}^{(m)} = \beta_i^2 K_{ik}^{(2)} + (-1)^m\frac{4}{3}\gamma^3 K_{ik}^{(5)}, \quad (142)$$

$$\mathfrak{K}_C = \sum_{l,n}\left(C_{2,ln}^{(1)} - \frac{4}{3}\gamma^3 C_{5,ln}\right) - \frac{4}{3}\gamma^3 C_{5,123}, \tag{143}$$

$$Q_3 = \sum_{j=1}^{2}\left(R_{2,j3}^{(1)} - \frac{4}{3}\gamma^3 R_{5,j3}\right) - \frac{4}{3}\gamma^3 R_{5,123}, \tag{144}$$

$$p_{ik} = K_{ik}^{(2)} - \frac{r_2}{r_1}K_{ik}^{(1)} + R_i^{(k)}, \; R_3^{(1)} = \sum_{j=1}^{2}R_{2,j3}, \; R_i^{(k)} = 0 \text{ if } i \neq 3, \, k \neq 1, \quad (145)$$

$$e_{i0} = -r_1 q_{i0}^{(2)} + K_{i0}^{(1)}\sum_{j=1}^{3}q_{j0}^{(2)} - 2r_1(\beta_i^2 K_{i1}^{(2)} + \zeta_{13} + \zeta_i) + 2K_{i0}^{(1)}\Psi, \tag{146}$$

$$e_{i1} = r_1(q_{i1}^{(1)} + Q_i) + K_{i1}^{(1)}\sum_{j=1}^{3}q_{j0}^{(2)}, \quad \Psi = \sum_{i=1}^{3}\beta_i^2 K_{i1}^{(2)} + 2\sum_{ln}\zeta_{ln}, \tag{147}$$

$$\zeta_{ln} = \beta_l^2 \beta_n K_l^{(1)} K_n^{(1)}\lambda_{I1}^{(0)}(\sigma_{ln}), \quad \zeta_1 = \zeta_{12}, \quad \zeta_3 = \zeta_{23}, \quad Q_1 = 0. \tag{148}$$

6.4 Asymptotic analysis

$$\tilde{E}_{11} = -\overline{r}_2 + \overline{r}_1\theta_{13}^{1/3} - \overline{r}_2\mu^{3(1+\alpha)/2}z'(\sigma_{12}) - 2\overline{r}_1\mu^{5(1+\alpha)/2}\lambda_{21}'(\sigma_{12})$$
$$+ 2\overline{r}_1\mu^{3+\alpha}\left(\left(\sigma\lambda_{21}(\sigma + \theta_{12}\sigma_{23})\right)'_\sigma\big|_{\sigma=\sigma_{12}} - \Lambda_1 z'(\sigma_{13})\right) + O_\lambda(\mu^{3+2\alpha}),$$

$$\tilde{E}_{12} = \overline{r}_1 - \overline{r}_2\theta_{13}^{1/3} - 2\overline{r}_1\mu^{3+\alpha}\left(\lambda_{21}(\sigma_{12})z'(\sigma_{23}) - \Lambda_1 z'(\sigma_{13})\right)$$
$$+ \overline{r}_2\mu^{(7+3\alpha)/2}z'(\sigma_{23}) + O_\lambda(\mu^4), \quad \overline{r}_1 = \sum_{i=1}^{3}\theta_{1i}^{1/3}, \quad \overline{r}_2 = \sum_{i=1}^{3}\theta_{i3}^{1/3}$$

$$\tilde{E}_{21} = \frac{7}{3} + \frac{7}{3}\mu^{3(1+\alpha)/2}z'(\sigma_{12}) - 4\mu^2\lambda_{21}(\sigma_{23}) + O_\lambda(\mu^{3+2\alpha}),$$

$$\tilde{E}_{22} = -\frac{7}{3}\overline{r}_1 + \frac{7}{3}\mu^{(3+\alpha)/2} + 4\mu^2\left(\lambda_{21}(\sigma_{23}) - \frac{5}{3}z'(\sigma_{23})\right) + O_\lambda(\mu^{3+\alpha}),$$

$$\tilde{F}_1 = -\frac{6}{7}\mu^2\Big\{\lambda_{21}(\sigma_{23})\left(1 + \mu^{(1+\alpha)/2}\right) + \mu^{(3+\alpha)/2}\left(2\lambda_{43}(\sigma_{23})\right.$$

$$-\frac{11}{7}\lambda_{21}(\sigma_{23})) + O_\lambda(\mu^2)\Big\}, \quad \tilde{F}_2 = 2\mu^2\Big\{\lambda_{21}(\sigma_{23}) - \frac{56}{3}\mu^{1+\alpha}\big(\lambda_{21}(\sigma_{13})$$

$$-\lambda_{21}(\sigma_{12})\lambda_{21}(\sigma_{23})\big) + \frac{28}{3}\mu^{(3+\alpha)/2}\big(\lambda_{43}(\sigma_{23}) - \lambda_{21}(\sigma_{23})\big) + O_\lambda(\mu^2)\Big\}.$$